

Removing an edge from a moderately balanced graph

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Abstract

A graph is *balanced* if the length of each of its chordless cycles is a multiple of 4. Conforti and Rao [CR92] conjecture that every balanced graph G contains an edge e such that $G \setminus e$ is balanced, or equivalently, every balanced graph contains an edge that is not the unique chord of a cycle. It is known that the conjecture holds for *totally balanced* (or *chordal bipartite*) graphs, which are balanced graphs that do not contain a hole of length greater than 4, but as noted in [CR92], it is not known whether the conjecture holds for *linear balanced* graphs, which are balanced graphs that do not contain a hole of length 4. We make progress on the conjecture and show that it holds for balanced graphs that do not contain a uniquely-chorded cycle with one and only one hole of length 4, which we call *moderately balanced* graphs, implying the conjecture holds for totally balanced graphs and linear balanced graphs.

1 Introduction

A graph is *balanced* if the length of each of its chordless cycles is a multiple of 4. (It follows that balanced graphs are bipartite.) In this work, we consider the following conjecture of Conforti and Rao [CR92].

Conjecture 1.1. *Every balanced graph G contains an edge e such that $G \setminus e$ is balanced.*

The conjecture was originally stated in terms of *balanced* matrices, and we briefly discuss this in Subsection 1.1. The conjecture is also mentioned in [CCR99] and [Cor01]. In fact, in [Cor01], Cornuéjols proposed 18 conjectures, including Conjecture 1.1. There are three other conjectures about balanced graphs; [CS08] give a counterexample to the first two and a proof to the third.

Conjecture 1.1 is clearly equivalent to:

Conjecture 1.2. *Every balanced graph contains an edge that is not the unique chord of a cycle.*

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It is known that the Conjecture 1.2 holds for *totally balanced* (or *chordal bipartite*) graphs, which are balanced graphs that do not contain a hole of length greater than 4, because every totally balanced graph contains a *bisimplicial edge*, see [GG78]. On the other hand, as noted in [CR92], it is not known if the conjecture holds for *linear balanced* graphs, which are balanced graphs that do not contain a hole of length 4.

We make progress on Conjecture 1.2 and show that it holds for balanced graphs that do not contain a uniquely-chorded cycle with one and only one hole of length 4, which we will call *moderately balanced* graphs.

Theorem 1.1. *Every moderately balanced graph G contains an edge that is not the unique chord of a cycle.*

Note that Theorem 1.1 implies that Conjecture 1.2 holds for totally balanced graphs and linear balanced graphs. In Subsection 1.2 we provide an outline of the proof of Theorem 1.1.

1.1 Balanced matrices

A *balanced* matrix is a $0, 1$ matrix that does not contain a square submatrix of odd order with two ones per row and per column. We can construct a balanced graph G from a balanced matrix A as follows. Construct the bipartite graph $G = (U, V, E)$, where the vertex set U contains a vertex u_i for each row i of A , the vertex set V contains a vertex v_j for each column j of A , and $u_i v_j$ is an edge in G if and only if $A_{ij} = 1$. In a similar spirit, we can construct a balanced matrix from a balanced graph. It follows that, in terms of balanced matrices, Conjecture 1.1 asserts that every balanced matrix contains an entry equal to 1, which upon setting to 0, gives a resulting matrix that is balanced.

Balanced matrices play an interesting role in combinatorial optimization. For example, the covering polytope $\{x : Ax \geq 1, 0 \leq x \leq 1\}$, packing polytope $\{x : Ax \leq 1, 0 \leq x \leq 1\}$, and partitioning polytope $\{x : Ax = 1, 0 \leq x \leq 1\}$ are integral if and only if A is balanced [Ber72]. See [CCV06] for a survey on balanced matrices.

1.2 Main definitions and an outline of the proof of Theorem 1.1

Let us introduce the main notions used in this work along with an outline of the proof of Theorem 1.1. The notation used in this subsection is standard, but if there is a need for clarification, see Subsection 1.3.

Definition 1.1. A vertex u in a graph G is a *minimal vertex* if there does not exist a vertex $w \in V(G)$ such that $N(w) \subsetneq N(u)$.

Definition 1.2. A collection of vertices U in a graph G is a *separator* for vertices $u_1 \in V(G)$ and $u_2 \in V(G)$ if $G \setminus U$ is disconnected and if u_1 and u_2 belong to different components in $G \setminus U$.

For a vertex u in a graph G , define the vertex sets

$$S(u) := N[N[u]] \quad \text{and} \quad \partial(u) := N(N[u]),$$

where $N(U)$, for $U \subseteq V(G)$, denotes the collection of vertices in $V(G) \setminus U$ that are adjacent to some vertex in U , and $N[U] := N(U) \cup U$.

Definition 1.3. A vertex u in a graph G is a *separating vertex* if there exist vertices $u_1, u_2 \in \partial(u)$ such that $S = S(u) \setminus \{u_1, u_2\}$ is a separator for u_1 and u_2 , and the components in $G \setminus S$ containing u_1 and u_2 are nontrivial.

Remark 1.1. Consider a separating vertex u , and suppose that $S = S(u) \setminus \{u_1, u_2\}$ is a separator for $u_1 \in \partial(u)$ and $u_2 \in \partial(u)$ such that u_1 and u_2 belong to nontrivial components of $G \setminus S$. Then $S(u)$ is also a separator, but it does not necessarily separate any vertices into nontrivial components.

In Section 3 we establish:

Theorem 1.2. *If u is a minimal vertex in a moderately balanced graph G , and u is incident to the unique chord of a uniquely-chorded cycle, then u is separating.*

Definition 1.4. A collection of vertices M in a graph G is a *minimal separator* for vertices $u_1 \in V(G)$ and $u_2 \in V(G)$ if M is a separator for u_1 and u_2 and if no strict subset of M is a separator for u_1 and u_2 .

Remark 1.2. Note that if M is a minimal separator for vertices u_1 and u_2 , then each vertex in M is adjacent to a vertex in the component of $G \setminus M$ containing u_1 and a vertex in the component containing u_2 .

And we introduce the following notion of a *corner*:

Definition 1.5. See Figure 1. A *corner* (M, B) is a subgraph in a bipartite graph G induced by vertex sets M and B , which satisfy the following properties:

1. M is a minimal separator for some vertex u_1 and some vertex u_2 .
2. B is the vertex set of the component in $G \setminus M$ that either contains u_1 or u_2 .
3. The vertex sets

$$\hat{B} := B \cap N(M) \quad \text{and} \quad \tilde{B} := \{u \in B : M \subseteq \partial(u)\}$$

are nonempty, partition B , and every minimal vertex in $\tilde{B}$ is separating.

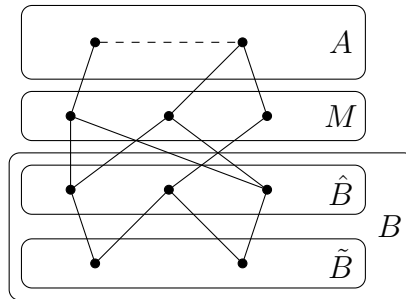


Figure 1: A corner (M, B) , where $G \setminus M$ has two components, and A and B are the vertex sets of the components.

Remark 1.3. From the third property of a corner (M, B) , it holds that M is a subset of one side of the bipartition, $\tilde{B}$ is a subset of the same side of the bipartition, and $\hat{B}$ is a subset of the opposite side of the bipartition.

Remark 1.4. In the definition of a corner (M, B) , the assumption that every minimal vertex in $\tilde{B}$ is separating ensures that $(M, B) \neq (\partial(u), N[u])$ for some vertex u .

Definition 1.6. A corner (M, B) is a *tight corner* if the distance between u_1 and u_2 is 2 for all $u_1, u_2 \in \tilde{B}$.

In Section 4 we show that if every minimal vertex in a bipartite graph G is separating, then G contains a tight corner, see Theorem 1.3 and Figure 2.

Theorem 1.3. *If every minimal vertex in a bipartite graph G is separating, then G contains a tight corner.*

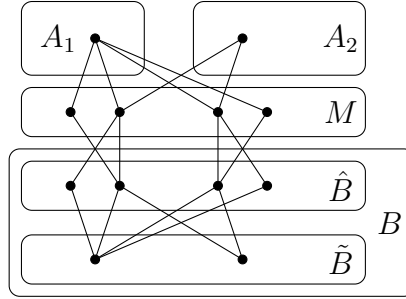


Figure 2: A bipartite graph G , where every minimal vertex is a separating vertex, and a tight corner (M, B) in G , where $G \setminus M$ has three components, and A_1 , A_2 , and B are the vertex sets of the components.

In Section 5 we show that:

Theorem 1.4. *If (M, B) is a tight corner in a moderately balanced graph G , then $E(\hat{B}, \tilde{B})$ contains an edge that is not the unique chord of a cycle in G .*

Theorem 1.1 follows from Theorem 1.2, Theorem 1.3, and Theorem 1.4:

Proof of Theorem 1.1. For the sake of contradiction, suppose that every edge in G is the unique chord of a cycle. Then it follows from Theorem 1.2 that every minimal vertex in G is a separating vertex, and hence by Theorem 1.3, it holds that G contains a tight corner (M, B) . Theorem 1.4 implies that $E(\hat{B}, \tilde{B})$ contains an edge that is not the unique chord of a cycle in G , a contradiction. $\square$

1.3 Notation

Let G be a graph. We write the vertex set and edge set of G as $V(G)$ and $E(G)$, respectively. For disjoint vertex sets A and B , let $E(A, B)$ denote the set of edges that are incident to a vertex in A and a vertex in B .

Inclusions. For sets X and Y , let $X \subseteq Y$ denote that X is a subset of Y , let $X \subsetneq Y$ denote that X is a strict subset of Y , and let $X \not\subseteq Y$ denote that X is not a subset of Y .

Induced graphs. For a vertex set U , let $G[U]$ denote the subgraph induced by U . We will write $G[V(G) \setminus U]$ as $G \setminus U$, and if $U = \{u\}$ for some vertex u , we will write $G \setminus \{u\}$ as $G \setminus u$. Similarly, for an edge set D , let $G \setminus D$ denote the graph obtained from G by removing the edges in D , and if $D = \{e\}$ for some edge e , we will write $G \setminus \{e\}$ as $G \setminus e$. And for subgraphs H_1 and H_2 , let $H_1 \cup H_2$ denote the subgraph $G[V(H_1) \cup V(H_2)]$.

Adjacencies. For a vertex u , let $N(u)$ denote the set of neighbors of u , and let $N[u] := N(u) \cup \{u\}$. For a vertex set A such that $u \notin A$, let $N_A(u)$ be the set of neighbors of u in A , and we will say u is *adjacent* to A if $N_A(u) \neq \emptyset$. For a vertex set U , let $N(U)$ denote the vertices in $V(G) \setminus U$ adjacent to U , and let $N[U] := N(U) \cup U$. And for a vertex set A such that $U \cap A = \emptyset$, let $N_A(U) = N(U) \cap A$.

Paths. We often write a path P with vertex set $\{x_1, \dots, x_n\}$, where $n \geq 1$, as $P = x_1x_2 \dots x_n$. For $i \leq j$, and vertices x_i and x_j in P , define the subpath $P(x_i, x_j) := x_ix_{i+1} \dots x_j$. We write $P = P(x_1, x_{i-1})P(x_i, x_j)P(x_{j+1}, x_n)$. When $n \geq 3$, we write $P(x_2, x_{n-1})$, which we call the *interior* of P , as $\tilde{P}$. Define the *length* of P to be the number of edges in P .

Direct connections. Let A , B , and C be disjoint vertex sets such that no vertex in A is adjacent to a vertex in B . A path $P = x_1x_2 \dots x_n$ *connects* A and B if x_1 is adjacent to A and x_n is adjacent to B . A path P that connects A and B is a *direction connection* between A and B if in the subgraph $G[A \cup B \cup P]$, no path that connects A and B is shorter than P . A direct connection P between A and B *avoids* C if $V(P) \cap C = \emptyset$.

Distances. We denote the distance between vertices u_1 and u_2 by $d(u_1, u_2)$. For a vertex u and a vertex set A , define $d(u, A) := \min_{a \in A} d(u, a)$.

2 Additional definitions

In this section we introduce some additional definitions that we will need.

2.1 3-wheels and 3-path configurations

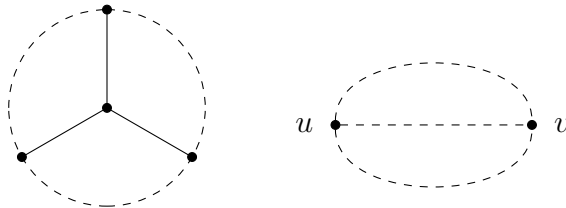


Figure 3: A 3-wheel and a 3-path configuration.

Definition 2.1. See Figure 3. A *3-wheel* is a graph that consists of a hole H and a vertex $u \notin V(H)$ such that u has exactly 3 neighbors in $V(H)$.

Definition 2.2. See Figure 3. A *3-path configuration* is a bipartite graph that consists of three vertex-disjoint paths P_1 , P_2 , and P_3 between nonadjacent vertices u and v such that u and v belong to opposite sides of the bipartition and $E(\tilde{P}_1, \tilde{P}_2) = E(\tilde{P}_2, \tilde{P}_3) = E(\tilde{P}_1, \tilde{P}_3) = \emptyset$.

Every 3-wheel and 3-path configuration contains a hole of length congruent to 2 modulo 4, and therefore balanced graphs do not contain 3-wheels or 3-path configurations.

2.2 Uniquely-chorded cycles

Definition 2.3. Let C be a cycle with unique chord e . A *uniquely-chorded cycle* (C, e) is the graph $C \cup e$.

For a uniquely-chorded cycle (C, e) , let $H_1(C, e)$ and $H_2(C, e)$ denote its holes. If $e = uv$ for some vertices u and v , we write the vertices in $H_1(C, e) \setminus v$ and $H_2(C, e) \setminus v$ that are adjacent to u as $v_1(C, e)$ and $v_2(C, e)$, respectively. And we define $u_1(C, e)$ and $u_2(C, e)$ analogously. Let $P_1(C, e)$ and $P_2(C, e)$ denote the paths $H_1(C, e) \setminus \{u, v\}$ and $H_2(C, e) \setminus \{u, v\}$, respectively. See Figure 4.

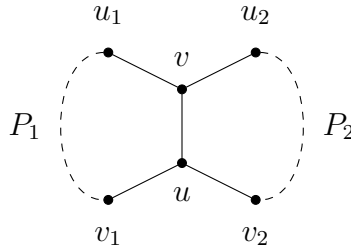


Figure 4: A uniquely-chorded cycle $(C, e = uv)$, where $u_1 = u_1(C, e)$, $v_1 = v_1(C, e)$, etc.

Definition 2.4. A *short* uniquely-chorded cycle contains two holes of length 4, a *long* uniquely-chorded cycle contains two holes of length greater than 8, and an *uneven* uniquely-chorded cycle contains one and only one hole of length 4.

2.3 Comparing vertices

Let u and w be vertices in a bipartite graph G that belong to the same side of the bipartition, and let A be a vertex set such that $N(u) \cap A \neq \emptyset$ and $N(w) \cap A \neq \emptyset$, implying A is a subset of the opposite side of the bipartition.

Definition 2.5. Vertex w *dominates* u if $N(u) \subseteq N(w)$, and w *strictly dominates* u if $N(u) \subsetneq N(w)$. Similarly w *dominates* u *relative to* A if $N_A(u) \subseteq N_A(w)$, and w *strictly dominates* u *relative to* A if $N_A(u) \subsetneq N_A(w)$.

Definition 2.6. Vertices u and w are *comparable* if w dominates u or if u dominates w . Similarly u and w are *comparable relative to* A if w dominates u relative to A or if u dominates w relative to A .

Definition 2.7. Vertices u and w are *twins* if $N(u) = N(w)$. Similarly u and w are *twins relative to A* if $N_A(u) = N_A(w)$.

Remark 2.1. Suppose (C, uv) is a uniquely-chorded cycle in G , and suppose w dominates u . Then $w \notin V(C)$ because no vertex in $V(C)$ dominates u . Therefore uv is still the unique chord of C in $G \setminus w$.

2.4 Pseudominimal separators

We introduce the notion of a *pseudominimal separator*, which is a relaxation of the notion of a minimal separator:

Definition 2.8. A collection of vertices M in a graph G is a *pseudominimal separator for A and B* if M is a separator for some vertices $u_1 \in V(G)$ and $u_2 \in V(G)$, A and B are vertex sets of components in $G \setminus M$ containing u_1 and u_2 , and every vertex in M is adjacent to A .

3 Separating vertices in uniquely-chorded cycles

In this section we prove Theorem 1.2, which asserts that a minimal vertex u in a moderately balanced graph G is separating if u is incident to the unique chord of a cycle. There are two cases to consider: u is incident to the unique chord of a short uniquely-chorded cycle, and u is incident to the unique chord of a long uniquely-chorded cycle.

The former case is straightforward:

Lemma 3.1. *If u is a minimal vertex in a balanced graph G , and u is incident to the unique chord of a short uniquely-chorded cycle, then u is separating.*

Proof. Suppose that (C, e) is a short uniquely-chorded cycle and $e = uv$, where v is some vertex. Observe that $S = S(u) \setminus \{u_1, u_2\}$ is a separator for $u_1 = u_1(C, e)$ and $u_2 = u_2(C, e)$ because if not, then there exists a direct connection P between u_1 and u_2 that avoids S , implying $(C, e) \cup P$ is a 3-wheel. By symmetry, it remains to verify that the component in $G \setminus S$ containing u_1 is nontrivial. Because u is minimal and incident to the unique chord of a short-uniquely chorded cycle, there exists a vertex $v' \notin S(u)$ that is adjacent to u_1 , implying the component in $G \setminus S$ containing u_1 is nontrivial. $\square$

Next we consider the case in which u is incident to the unique chord of a long uniquely-chorded cycle:

Lemma 3.2. *If u is a vertex in a moderately balanced graph G , and u is incident to a unique chord of a long uniquely-chorded cycle, then u is separating.*

Let us introduce some notation and definitions that we will just use for the remainder of the subsection. For a long uniquely-chorded cycle $(C, e = uv)$, let us simplify notation and write $u_1 = u_1(C, e)$, $u_2 = u_2(C, e)$, etc.

Let u_3 denote the vertex adjacent to v_1 in $H_1 \setminus u$, and let u_4 denote the vertex adjacent to v_2 in $H_2 \setminus u$. See Figure 5.

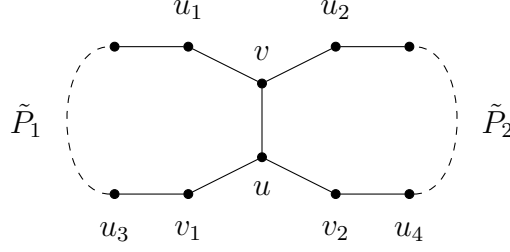


Figure 5: A long uniquely-chorded cycle $(C, e = uv)$ containing vertices u_3 and u_4 .

For a vertex $x \notin V(C)$ that is adjacent to $V(H_1)$, define $P_{1u}(x)$ to be the shortest path in $H_1 \setminus e$ between u and a neighbor of x in $V(H_1)$, and similarly define $P_{1v}(x)$ to be the shortest path in $H_1 \setminus e$ between v and a neighbor of x in $V(H_1)$. For a vertex $x \notin V(C)$ adjacent to $V(H_2)$, define $P_{2u}(x)$ and $P_{2v}(x)$ analogously. See Figure 6.

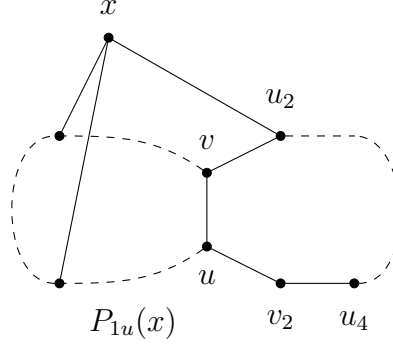


Figure 6: A vertex $x \notin V(C)$ adjacent $V(H_1)$ along with path $P_{1u}(x)$. Note that in this figure, but not in general, $P_{2v}(x) = u_2v$ and $P_{2u}(x) = P_2u$.

For a long uniquely-chorded cycle (C, e) in a balanced graph G , we establish properties of direct connections $P = x_1x_2 \dots x_n$ between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoid $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$. See Figure 7 for an example of such a direct connection.

Lemma 3.3. *Suppose (C, e) is a long uniquely-chorded cycle in a balanced graph G , and $P = x_1x_2 \dots x_n$ is a direct connection between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoids the vertex set $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$. Then*

1. *the length of P is at least 1,*
2. *u_1 is adjacent to $V(P(x_2, x_n))$, and*
3. *u_2 is adjacent to $V(P(x_1, x_{n-1}))$.*

Proof. Let $P = x_1x_2 \dots x_n$ be a direct connection between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoids $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$. Suppose that P is either of length 0, or P is of length at least 1,

and without loss of generality, u_1 is not adjacent to $V(P(x_2, x_n))$. We consider a few cases, and in each case, we construct a 3-path configuration:

Case 1. Suppose that x_1 has at least 2 neighbors in $V(H_1) \setminus \{u, v, v_1\}$.

Subcase 1.1. Suppose that x_1 and u do not belong to the same side of the bipartition. Then path $x_1P_{1u}(x_1)$, path $x_1P_{1v}(x_1)u$, and path $PP_{2u}(x_n)$ comprise a 3-path configuration with endpoints x_1 and u .

Subcase 1.2. Suppose that x_1 and u belong to the same side of the bipartition. If u_2 is not adjacent to P , then path $x_1P_{1v}(x_1)$, path $x_1P_{1u}(x_1)v$, and path $PP_{2v}(x_n)$ comprise a 3-path configuration with endpoints x_1 and v . Suppose u_2 is adjacent to P . Let $P_3 = P(x_1, x_j)$, where j is the smallest index for which u_2 adjacent to x_j . Then path $x_1P_{1v}(x_1)$, path $x_1P_{1u}(x_1)v$, and path P_3u_2v comprise a 3-path configuration with endpoints x_1 and v .

Case 2. Suppose x_1 is adjacent to one and only vertex y in $V(H_1) \setminus \{u, v, v_1, u_1\}$.

Subcase 2.1. Suppose that y and u do not belong to the same side of the bipartition. Then path $P_{1u}(x_1)$, path $P_{1v}(x_1)u$, and path $yPP_{2u}(x_n)$ comprise a 3-path configuration with endpoints y and u .

Subcase 2.2. Suppose that y and u belong to the same side of the bipartition. If u_2 is not adjacent to P , then path $P_{1v}(x_1)$, path $P_{1u}(x_1)v$, and path $yPP_{2v}(x_n)$ comprise a 3-path configuration with endpoints y and v . Suppose u_2 is adjacent to P . Let $P_3 = P(x_1, x_j)$, where j is the smallest index for which u_2 adjacent to x_j . Then path $P_{1v}(x_1)$, path $P_{1u}(x_1)v$, and path yP_3u_2v comprise a 3-path configuration with endpoints y and v . $\square$

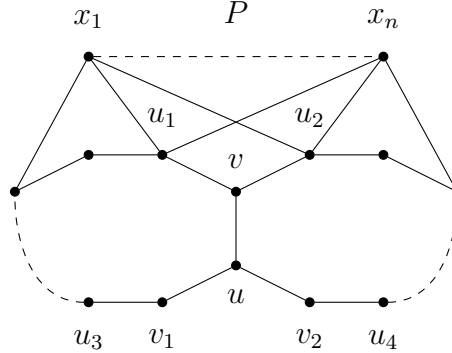


Figure 7: A long uniquely-chorded cycle (C, e) in a balanced graph along with a direct connection $P = x_1x_2 \dots x_n$ between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoids $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$. Note that the length of P must be congruent to 2 modulo 4.

Next we show that if G is moderately balanced, then G does not contain such direct connections. (Note that the graph in Figure 7 contains an uneven uniquely-chorded cycle.)

Lemma 3.4. *Suppose (C, e) is a long uniquely-chorded cycle in a moderately balanced graph G . Then G does not contain a direct connections between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoids $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$.*

Proof. Suppose that there exists a direct connection $P = x_1, \dots, x_n$ between $V(\tilde{P}_1)$ and $V(\tilde{P}_2)$ that avoids $N[\{u, v, v_1, v_2\}] \setminus \{u_3, u_4\}$. From Lemma 3.3, the length of P is at least 1, vertex u_1 is adjacent to $V(P(x_2, x_n))$, and u_2 is adjacent to $V(P(x_1, x_{n-1}))$.

Suppose that u_1 and u_2 are adjacent to $V(\tilde{P})$. Let $\hat{P}$ be a subpath in $\tilde{P}$ that contains one and only one neighbor of u_1 and one and only one neighbor of u_2 . Then $(C, e) \cup \hat{P}$ is a 3-wheel. Hence at least one of the following holds: u_2 is adjacent to x_1 , or u_1 is adjacent to x_n . Without loss of generality, suppose that u_2 is adjacent to x_1 .

It holds that x_1 is not adjacent to u_1 because if x_1 is adjacent to u_1 , then vu_2 is the unique chord of the uneven uniquely-chorded cycle with induced holes $x_1u_1vu_2x_1$ and H_2 . Recall that u_1 is adjacent to $V(P(x_2, x_n))$. Let j be the smallest index such that u_1 is adjacent to x_j . If x_j is adjacent to u_2 , then vu_1 is the unique chord of the uneven uniquely-chorded cycle with induced holes $u_1vu_2x_ju_1$ and H_1 . Hence x_j is not adjacent to u_2 . Thus $P_1(x_1, v) \cup P(x_1, x_j) \cup u_2$ is a 3-path configuration with endpoints x_1 and u_1 . $\square$

We are now prepared to establish Lemma 3.2:

Proof of Lemma 3.2. It follows from Lemma 3.4 that $S = S(u) \setminus \{u_3, u_4\}$ is a separator for u_3 and u_4 . To establish u is separating, it remains, by symmetry, to verify that u_3 belongs to a nontrivial component in $G \setminus S$. Let v_5 be the vertex adjacent to u_3 in $H_1 \setminus v_1$. Then v_5 belongs to the same component in $G \setminus S$ as u_3 because v_5 is not adjacent to u . Thus u is separating. $\square$

Theorem 1.2 follows from Lemma 3.1 and Lemma 3.2.

4 Tight corners in bipartite graphs

In this section we establish:

Theorem 4.1. *Suppose that M is a pseudominimal separator for vertex sets A and B in a bipartite graph G , every minimal vertex in B is a separating vertex, and B contains a vertex u such that $d(u, A) \geq 3$. Then G contains a tight corner (M', B') such that $B' \subseteq B$.*

Theorem 1.3 immediately follows from Theorem 4.1 as follows. (We will also use Theorem 4.1 in Section 5.)

Proof of Theorem 1.3. Because every minimal vertex in G is separating, it holds that for a minimal vertex u , there exists a vertex v such that $d(u, v) \geq 3$. Let $A = \{u\}$, $M = N(u)$, and B be the vertex set of the component in $G \setminus M$ containing v . Then M is a pseudominimal separator for A and B , and it holds that $d(u, A) \geq 3$. Theorem 4.1 implies G contains a tight corner. $\square$

Let us introduce some notation and definitions that we will just use for the remainder of this subsection. For a collection of vertices A in a bipartite graph G such that $G[A]$ is connected and a vertex u such that $d(u, A) \geq 3$, let A_u denote the vertex set of the component in $G \setminus S(u)$ containing A , and let M_u denote the vertices in $V(G) \setminus A_u$ adjacent to A_u . (Note that we did not suppose that u is separating, and therefore A_u might be the only component in $G \setminus S(u)$.) It is straightforward to see that $M_u \subseteq \partial(u)$, vertex set A_u is

the vertex set of the component in $G \setminus M_u$ containing A , and M_u is a minimal separator for u and a , where a is any vertex in A_u . Let B_u be the vertex set of the component in $G \setminus M_u$ containing u .

Lemma 4.1. *Suppose that M is a pseudominimal separator for vertex sets A and B in a bipartite graph G , and B contains a vertex u such that $d(u, A) \geq 3$. Then $B_u \subseteq B$.*

Proof. Suppose that $B_u \setminus B$ contains a vertex x . Then – because M is a pseudominimal separator for A and B – either $x \in M$ or x belongs to a component of $G \setminus M$ other than B . If $x \in M$, then x is adjacent to A , but no vertex in B_u is adjacent to A because M_u is a minimal separator for B_u and $A_u \supseteq A$. It follows that x must belong to a component of $G \setminus M$ other than B , but then every path P in $G[B_u]$ between x and u contains a vertex in M because M is a pseudominimal separator for A and B . $\square$

We establish a weak version of Theorem 4.1, which asserts the existence of a corner (not necessarily a tight corner):

Lemma 4.2. *Suppose that M is a pseudominimal separator for vertex sets A and B in a bipartite graph G , every minimal vertex in B is a separating vertex, and B contains a vertex u such that $d(u, A) \geq 3$. Then G contains a corner (M', B') such that $B' \subseteq B$.*

Proof. Suppose that $G = (X, Y, E)$, namely the bipartition of $V(G)$ is given by $\{X, Y\}$. Let

$$\begin{aligned} x \in \operatorname{argmax}_u & |A_u| \\ \text{s.t.} & d(u, A) \geq 3 \\ & u \in B. \end{aligned}$$

Without loss of generality, suppose that $x \in X$. Let us show that $(M', B') = (M_x, B_x)$ is a corner. (Note that $B_x \subseteq B$ by Lemma 4.1.)

Recall that M_x is a minimal separator for x and a , where a is any vertex in A_x , and B_x is the component in $G \setminus M_x$ containing x . Therefore the first two properties of a corner hold. It remains to verify the third property. First note that $B_x \cap Y \neq \emptyset$ and $B_x \cap X \neq \emptyset$ because $N[x] \subseteq B_x$. Now observe that:

1. For $y \in B_x \cap Y$, it holds that y is adjacent to M_x because if not, then $d(y, M_x) \geq 3$ because $M_x \subseteq \partial(x) \subseteq X$, and hence $M_x \cup A_x \subseteq A_y$, implying that $|A_x| < |A_y|$.
2. For $z \in B_x \cap X$, it holds that $M_x \subseteq \partial(z)$ because if not, then there exists a vertex $v \in M_x \setminus \partial(z)$, and it follows that $A_x \cup \{v\} \subseteq A_z$, implying that $|A_x| < |A_z|$.
3. Every minimal vertex in $B_x \cap X$ is separating because $B_x \cap X \subseteq B$.

Thus the third property of a corner holds, completing the proof. $\square$

If (M, B) is a corner, but not a tight corner, then we can construct a “smaller” corner:

Lemma 4.3. *Suppose (M, B) is a corner, but not a tight corner, in a bipartite graph G . Then G contains a corner (M', B') such that $B' \subsetneq B$.*

Proof. Suppose that $G = (X, Y, E)$, namely the bipartition of $V(G)$ is given by $\{X, Y\}$. Without loss of generality, suppose that $M \cup \tilde{B} \subseteq X$. Because (M, B) is not a tight corner, there exist vertices $x_1, x_2 \in \tilde{B}$ such that $d(x_1, x_2) \geq 4$. For each vertex $u \in \tilde{B}$ such that $d(u, x_2) \geq 4$, let A_{Bu} be the vertex set of the component in $G[B \setminus S(u)]$ that contains $N[x_2]$. Let

$$\begin{aligned} x \in \operatorname{argmax}_u \quad & |A_{Bu}| \\ \text{s.t.} \quad & d(u, N[x_2]) \geq 3 \\ & u \in \tilde{B}. \end{aligned}$$

Let $\hat{M}$ denote the vertices in $B \setminus A_{Bu}$ adjacent to A_{Bu} , and let B' be the vertex set of the component in $G[B] \setminus \hat{M}$ containing x . Note that in $G[B]$, it holds that $\hat{M}$ is a minimal separator for x and x_2 . Also note that B' and A_{Bu} are vertex sets of components in $G[B] \setminus \hat{M}$. Let $M' = M \cup \hat{M}$, and let us show that (M', B') is a corner. (And this will complete the proof because clearly $B' \subsetneq B$.)

Let us verify the first property of a corner. Clearly M' is a separator for x and x_2 . We show that x and x_2 are connected in $G \setminus \tilde{M}$, where $\tilde{M} \subsetneq M'$, implying M' is a minimal separator for x and x_2 . Let $m \in M' \setminus \tilde{M}$, and observe that if $m \in \hat{M}$, then x and x_2 are connected in $G \setminus \tilde{M}$ because in $G[B]$, the vertex set $\hat{M}$ is a minimal separator that separates x and x_2 . Suppose that $m \in M$. Because (M, B) is a corner, it holds that $m \in \partial(x)$ and $m \in \partial(x_2)$, and since $N[x] \subseteq B'$ and $N[x_2] \subseteq A_{Bx}$, it follows that there exist vertices $y_1 \in N(x)$ and $y_2 \in N(x_2)$ such that xy_1m and x_2y_2m are paths in $G \setminus \tilde{M}$, implying x and x_2 are connected.

Clearly the second property of a corner holds, and therefore let us verify the third property. Note that $B' \cap X \neq \emptyset$ and $B' \cap Y \neq \emptyset$ because $N[x] \subseteq B'$. Now observe that:

1. Each $y \in B' \cap Y$ is adjacent to M because (M, B) is a corner, and hence each $y \in B' \cap Y$ is adjacent to M' .
2. For $z \in B' \cap X$, it holds that $z \in \tilde{B}$ and $N[z] \subseteq B'$, and hence $M \subseteq \partial(z)$ because (M, B) is a corner. It also holds that $\hat{M} \subseteq \partial(z)$ (otherwise, from an argument similar to the argument given in the proof of Lemma 4.2, it would follow that $|A_{Bx}| < |A_{Bz}|$). Thus $M' = M \cup \hat{M} \subseteq \partial(z)$.
3. Every minimal vertex in $B' \cap X$ is separating because $B' \cap X \subseteq \tilde{B}$.

Thus the third property of a corner holds, completing the proof. $\square$

It is now straightforward to prove Theorem 4.1:

Proof of Theorem 4.1. From Lemma 4.2, it holds that G contains a corner (M_1, B_1) such that $B_1 \subseteq B$. If (M_1, B_1) is a tight corner, then the desired result follows. Otherwise, if (M_1, B_1) is not a tight corner, then Lemma 4.3 yields that there exists a corner (M_2, B_2) such that $B_2 \subsetneq B_1$. Iterating this argument yields the desired result. $\square$

5 Uniquely-chorded cycles and tight corners

In this section we establish Theorem 1.4. In Subsections 5.1-5.3 we establish preliminaries; in Subsection 5.1 we define three vertex-induced subgraphs that a corner in a moderately balanced graph does not contain, in Subsection 5.2 we introduce some additional definitions and establish related properties, and in Subsection 5.3 we define three types of tight corners, namely type-1, type-2, and type-3 tight corners. In Subsection 5.4 and 5.5 we show that every tight corner in a moderately balanced graph is one of these three types, and in Subsection 5.6, we use this result to prove Theorem 1.4.

5.1 Type-a, type-b, and type-c subgraphs

Let (M, B) be a corner in a bipartite graph G , and let $m_1, m_2 \in M$, let $y_1, y_2, y_3 \in \hat{B}$, and let $x_1, x_2 \in \tilde{B}$. We define three subgraphs of (M, B) that can be induced by these, or a subset of these, vertices:

Definition 5.1. *Type-a, type-b, and type-c subgraphs* are vertex-induced subgraphs defined, for convenience, in Figure 8.

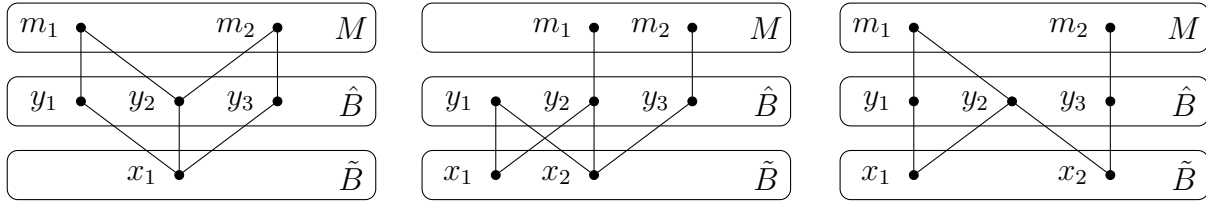


Figure 8: Type-a, type-b, and type-c subgraphs of a corner (M, B) .

Note that if (M, B) contains a type-a subgraph, then G contains a 3-wheel, implying G is not balanced. And if (M, B) contains a type-b or type-c subgraph, then G contains an uneven uniquely-chorded cycle, implying G is not moderately balanced.

5.2 Additional definitions and related properties

For a corner (M, B) , define the vertex sets

$$B^* := \{y \in \hat{B} : \tilde{B} \subseteq N(y)\}, \quad \hat{M} := N_M(\hat{B} \setminus B^*), \quad \text{and} \quad \tilde{M} := M \setminus \hat{M}.$$

And we establish a few related properties:

Remark 5.1. If (M, B) is a tight corner in a bipartite graph G , then $\hat{B} \setminus B^* \neq \emptyset$ because if not, then no vertex in $\tilde{B}$ is a separating vertex. It follows that $\hat{M} \neq \emptyset$.

Lemma 5.1. *If (M, B) is a tight corner in a balanced graph G , then for every vertex $x \in M$, there exists a vertex $y \in B^*$ such that $x \in N(y)$.*

Proof. First we establish the following claim.

Claim 5.1. *If vertex $v \notin S(u)$ is adjacent to $\partial(u)$ for some vertex u , then there exists a vertex $v' \in N(u)$ such that v' dominates v relative to $\partial(u)$.*

Proof of Claim 5.1. The claim is trivial when v has one and only one neighbor in $\partial(u)$. Accordingly, suppose v has at least 2 neighbors in $\partial(u)$.

For the sake of contradiction, suppose that there exist vertices $v_1 \in N(u)$ and $v_2 \in N(u)$ that are not comparable relative to $\partial(u) \cap N(v)$, namely v_1 has a neighbor $u_1 \in (\partial(u) \cap N(v)) \setminus N(v_2)$, and v_2 has a neighbor $u_2 \in (\partial(u) \cap N(v)) \setminus N(v_1)$. Then $uv_1u_1vu_2v_2u$ is a 6-hole. It follows that all vertices in $N(u)$ that are adjacent to $\partial(u) \cap N(v)$ are comparable, and it is straightforward to see that this implies the desired result. $\square$

First we show that $B^* \neq \emptyset$. Suppose that $B^* = \emptyset$. Let $y_1 \in \hat{B}$ such that, amongst vertices in $\hat{B}$, vertex y_1 is a vertex with the largest number of neighbors in $\tilde{B}$. By supposition, there exists a vertex $x \in \tilde{B} \setminus N(y_1)$. Notice that $\tilde{B} \setminus \{x\} \subseteq \partial(x)$ because (M, B) is a tight corner, and so it follows that $N_{\partial(x)}(y_1) \supseteq N_{\tilde{B}}(y_1) \neq \emptyset$. From Claim 5.1 with $v = y_1$ and $u = x$, there exists a vertex $y_2 \in N(x) \subseteq \hat{B}$ such that y_2 dominates y_1 relative to $\partial(x) \supseteq \tilde{B} \setminus \{x\}$, and hence $N_{\tilde{B}}(y_1) \cup \{x\} \subseteq N_{\tilde{B}}(y_2)$, a contradiction.

For the sake of contradiction, suppose that there exists a vertex $m \in M$ that is not adjacent to B^* . Let $y_1 \in B^*$. Because (M, B) is a corner, we have $\partial(m) \supseteq \tilde{B} = N_{\tilde{B}}(y_1)$, and hence $N_{\partial(m)}(y_1) \supseteq N_{\tilde{B}}(y_1) \neq \emptyset$. From Claim 5.1 with $v = y_1$ and $u = m$, there exists a vertex $y_2 \in N(m)$ such that y_2 dominates y_1 relative to $\partial(m) \supseteq \tilde{B}$. It follows that $y_2 \in B^*$, but m is adjacent to y_2 , a contradiction. $\square$

Remark 5.2. From Lemma 5.1, if (M, B) is a tight corner in a balanced graph G , then $B^* \neq \emptyset$.

Lemma 5.2. *Suppose that (M, B) is a tight corner in a moderately balanced graph G , and let $xy \in E(\tilde{B}, \hat{B})$ such that $x \in \tilde{B}$ and $y \in \hat{B}$. If xy is the unique chord of a uniquely-chorded cycle, then not every vertex in $N(x) \setminus \{y\}$ dominates y relative to $\tilde{B}$.*

Proof. Suppose that $xy = e$ and that e is the unique chord of the uniquely-chorded cycle (C, e) . For the sake of contradiction, suppose that every vertex in $N(x) \setminus \{y\}$ dominates y relative to $\tilde{B}$. Then $x_1 = x_1(C, e) \in M$ and $x_2 = x_2(C, e) \in M$ because, by supposition, $y_1 = y_1(C, e) \in N(x)$ and $y_2 = y_2 \in N(x)$ dominate y relative to $\tilde{B}$. If (C, e) is a short uniquely-chorded cycle, then the vertices x_1, x_2, y_1, y, y_2 , and x induce a type-a subgraph. Accordingly, (C, e) must be a long uniquely-chorded cycle. Let x_3 denote the neighbor of y_1 in $H_1(C, e) \setminus x$, and let x_4 denote the neighbor of y_2 in $H_2(C, e) \setminus x$. Then $x_3, x_4 \in M$ (for the same reason that $x_1, x_2 \in M$), contradicting Lemma 3.4. $\square$

5.3 Type-1, type-2, and type-3 tight corners

We define type-1, type-2, and type-3 tight corners as follows:

Definition 5.2. A tight corner (M, B) is a *type-1 tight corner* if there exists a partition $\{B_1^*, B_2^*\}$ of B^* such that $\hat{M} \cup N_M(B_2^*) \subseteq N(y)$ for all $y \in B_1^*$.

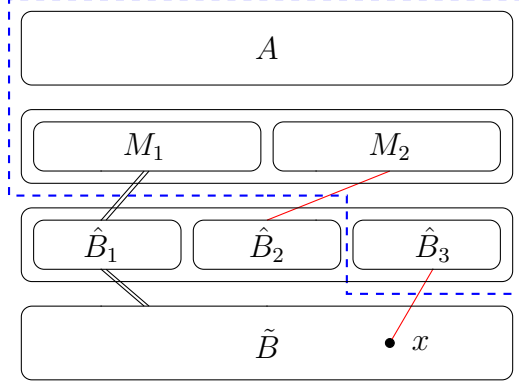


Figure 9: A type-2 tight corner (M, B) , where parallel lines joining vertex sets denote that the vertex sets induce a complete bipartite subgraph, red lines joining vertex sets (or a vertex and a vertex set) denote adjacencies that cannot occur, and the vertex sets in the region within the blue dashed line induce a connected subgraph.

Definition 5.3. See Figure 9. A tight corner (M, B) is a *type-2 tight corner* if there exists a partition $\{\hat{B}_1, \hat{B}_2, \hat{B}_3\}$ of $\hat{B}$ and a partition $\{M_1, M_2\}$ of M that satisfy the following four properties:

1. For every vertex $y \in \hat{B}_1$, it holds that $\tilde{B} \cup M_1 \subseteq N(y)$.
2. The graph $G[A \cup M_2 \cup \hat{B}_3]$ is connected.
3. No vertex in $\hat{B}_2$ is adjacent to M_2 .
4. There exists a vertex $x \in \tilde{B}$ that is not adjacent to $\hat{B}_3$.

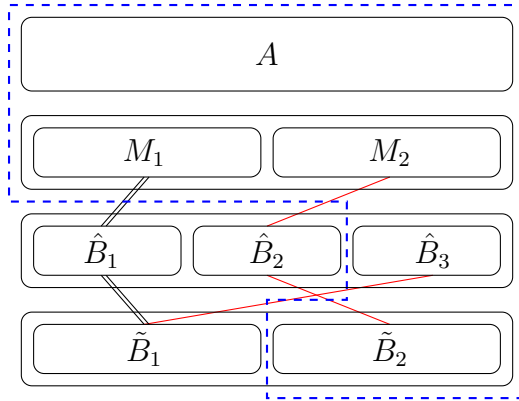


Figure 10: A type-3 tight corner (M, B) , where parallel lines joining vertex sets denote that the vertex sets induce a complete bipartite subgraph, red lines joining vertex sets denote adjacencies that cannot occur, and the vertex sets in the region within the blue dashed line induce a connected subgraph.

Definition 5.4. See Figure 10. A tight corner (M, B) is a *type-3 tight corner* if there exist partitions $\{\tilde{B}_1, \tilde{B}_2\}$, $\{\hat{B}_1, \hat{B}_2, \hat{B}_3\}$, and $\{M_1, M_2\}$ of $\tilde{B}$, $\hat{B}$, and M , respectively, that satisfy the following four properties:

1. For every vertex $y \in \hat{B}_1$, it holds that $M_1 \cup \tilde{B}_1 \subseteq N(y)$.
2. The graph $G[A \cup M_2 \cup \hat{B}_3 \cup \tilde{B}_2]$ is connected.
3. No vertex in $\hat{B}_2$ is adjacent to $\tilde{B}_2 \cup M_2$.
4. No vertex in $\tilde{B}_1$ is adjacent to $\hat{B}_3$.

5.4 Type-2 tight corners

Let (M, B) be a tight corner in a moderately balanced graph G , and suppose that (M, B) does not contain a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. In this subsection we show that such a tight corner, under a certain condition, is a type-2 tight corner, see Theorem 5.1 below.

First let us establish some preliminaries.

Lemma 5.3. *Let (M, B) be a tight corner in a moderately balanced graph G , and suppose that (M, B) does not contain a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Then there exist nonempty vertex sets $U, W \subseteq B^*$ such that*

1. *every two vertices in U are twins if $|U| > 1$,*
2. *every two vertices in W are twins if $|W| > 1$,*
3. *each $u \in U$ and each $w \in W$ are incomparable relative to $\hat{M}$, and*
4. *$\hat{M} \subseteq N(U) \cup N(W)$.*

And if $D = B^* \setminus (U \cup W) \neq \emptyset$, then for all $y \in D$, it holds that $N_M(y) \subseteq N(U) \cap N(W)$.

Proof. From the supposition that there does not exist a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$, there exist vertices $u \in B^*$ and $w \in B^*$ that are not comparable relative to $\hat{M}$, implying u and w are not comparable relative to M .

Let us establish the following claim:

Claim 5.2. *If vertices $y_1 \in B^*$ and $y_2 \in B^*$ are not comparable relative to M , then $\hat{M} \subseteq N(y_1) \cup N(y_2)$.*

Proof of Claim 5.2. Because y_1 and y_2 are not comparable relative to M , there exists a vertex $m_1 \in N_M(y_1) \setminus N(y_2)$ and a vertex $m_2 \in N_M(y_2) \setminus N(y_1)$. For the sake of contradiction, suppose that there exists a vertex $\hat{m} \in \hat{M} \setminus (N(y_1) \cup N(y_2))$. Then there exists a vertex $y \in N_{\hat{B} \setminus B^*}(\hat{m})$, a vertex $x_1 \in \tilde{B} \setminus N(y)$, and a vertex $x_2 \in N_{\tilde{B}}(y)$. If m_1 is not adjacent to y , then $m_1, \hat{m}, y_2, y_1, y, x_1$, and x_2 induce a type-b subgraph, and so m_1 must be adjacent to y . A similar argument yields that m_2 must be adjacent to y . Then m_1, m_2, y_1, y, y_2 , and x_2 induce a type-a subgraph, a contradiction. $\square$

Claim 5.2 implies that $\hat{M} \subseteq N(u) \cup N(w)$. Suppose that there exists a vertex $y \in B^* \setminus \{u, w\}$, and let us further suppose that $N_M(y) \not\subseteq N(u) \cap N(w)$. To establish the desired result, it is sufficient to show that y is a twin of u or that y is a twin of w , and we carry out this task in three steps:

Step 1. First we show that y is not comparable to at least one of u or w relative to M . Suppose that y is comparable to u and w relative to M . If y dominates u relative to M , and y dominates w relative to M , then $\hat{M} \subseteq N_M(u) \cup N_M(w) \subseteq N(y)$, a contradiction. Because u and w are not comparable relative to M , it is not possible for w to dominate y relative to M and for y to dominate u relative to M . (Similarly, it is not possible for u to dominate y relative to M and for y to dominate w relative to M .) Lastly, if u and w dominate y relative to M , then $N_M(y) \subseteq N(u) \cap N(w)$, a contradiction. Thus y is not comparable to at least one of u or w relative to M .

Without loss of generality, suppose that y is not comparable to u relative to M . In the remainder of the proof, we show that y and w are twins. To this end, it is sufficient to show that $N_M(y) = N_M(w)$.

Step 2. Let us show that $N_{\hat{M}}(y) = N_{\hat{M}}(w)$. Claim 5.2 implies that $\hat{M} \subseteq N(u) \cup N(y)$, and hence y is adjacent to every vertex in $N_{\hat{M}}(w) \setminus N(u)$. Let $m_1 \in N_{\hat{M}}(w) \setminus N(u)$. It remains to show that y is not adjacent to $N_{\hat{M}}(u) \setminus N(w)$ and that y is adjacent to every vertex in $N_{\hat{M}}(u) \cap N_{\hat{M}}(w)$. If y is adjacent to $m_2 \in N_{\hat{M}}(u) \setminus N(w)$, then for any $x \in \tilde{B}$, the vertices m_2, m_1, u, y, w , and x induce a type-a subgraph. Thus y is not adjacent to $N_{\hat{M}}(u) \setminus N(w)$. Now observe that y is adjacent to every vertex in $N_{\hat{M}}(u) \cap N_{\hat{M}}(w)$ because otherwise, there exists a vertex $m_3 \in N_M(u) \cap N_M(w)$ that is not adjacent to y , implying for any vertex $x \in \tilde{B}$, the vertices m_3, m_1, u, w, y , and x induce a type-a subgraph.

Step 3. Finally, we show that $N_{\tilde{M}}(y) = N_{\tilde{M}}(w)$. Suppose that $N_{\tilde{M}}(y) \neq N_{\tilde{M}}(w)$. Let us further suppose that there exists a vertex $m_1 \in N_{\tilde{M}}(w) \setminus N(y)$. And let vertex $m_2 \in N_{\hat{M}}(u) \setminus N(w) = N_{\hat{M}}(u) \setminus N(y)$, vertex $y_1 \in N_{\tilde{B} \setminus B^*}(m_2)$, vertex $x_1 \in \tilde{B} \setminus N(y_1)$, and vertex $x_2 \in N_{\tilde{B}}(y_1)$. Then vertices m_1, m_2, y, w, y_1, x_1 , and x_2 induce a type-b subgraph. And there does not a vertex in $N_{\tilde{M}}(y) \setminus N(w)$ from a similar argument. Thus $N_{\tilde{M}}(y) = N_{\tilde{M}}(w)$. $\square$

Let us introduce some notation and definitions that we will just use for the remainder of this subsection. Let $U, W \subseteq B^*$ be as given in the setup of Lemma 5.3, and let $D := B^* \setminus (U \cup W)$. Define $\hat{M}_{u \setminus w}$ to be the set of vertices in $\hat{M}$ that are adjacent to U but that are not adjacent to W , and define $\hat{M}_{w \setminus u}$ to be the set of vertices in $\hat{M}$ that are adjacent to W but that are not adjacent to U . Note that $\hat{M}_{u \setminus w} \neq \emptyset$ and $\hat{M}_{w \setminus u} \neq \emptyset$. Also define $\hat{B}_{u \setminus w}$ to be the set of vertices in $\hat{B} \setminus B^*$ that are adjacent to $\hat{M}_{u \setminus w}$, and define $\hat{B}_{w \setminus u}$ to be the set of vertices in $\hat{B} \setminus B^*$ that are adjacent to $\hat{M}_{w \setminus u}$. Note that $\hat{B}_{u \setminus w} \neq \emptyset$ and $\hat{B}_{w \setminus u} \neq \emptyset$.

Lemma 5.4. *Let (M, B) be a tight corner in a moderately balanced graph G , and suppose that (M, B) does not contain a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Then there exists a vertex $x \in \tilde{B}$ that is not adjacent to $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$.*

Proof. First we show that $\hat{B}_{u \setminus w} \cap \hat{B}_{w \setminus u} = \emptyset$. Suppose that there exists a vertex $y \in \hat{B}_{u \setminus w} \cap \hat{B}_{w \setminus u}$. Then y is adjacent to some vertex $m_1 \in \hat{M}_{u \setminus w}$ and some vertex $m_2 \in \hat{M}_{w \setminus u}$. Then for $x \in N_{\tilde{B}}(y)$, it holds that vertices m_1, m_2, u, y, w , and x induce a type-a subgraph.

Next we show that there exists a vertex in $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$ that dominates every other vertex in $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$ relative to $\tilde{B}$. Let $y_1, y_2 \in \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$. It is sufficient to show that y_1 and y_2 are comparable relative to $\tilde{B}$. Suppose that y_1 and y_2 are not comparable relative to $\tilde{B}$, implying there exist vertices $x_1 \in N_{\tilde{B}}(y_1) \setminus N(y_2)$ and $x_2 \in N_{\tilde{B}}(y_2) \setminus N(y_1)$.

Case 1. Suppose that $y_1 \in \hat{B}_{u \setminus w}$ and $y_2 \in \hat{B}_{w \setminus u}$. Let $m_1 \in \hat{M}_{u \setminus w}$ and $m_2 \in \hat{M}_{w \setminus u}$ be adjacent to y_1 and y_2 , respectively. Then vertices $m_1, m_2, y_1, u, y_2, x_1$, and x_2 induce a type-c subgraph.

Case 2. Suppose that $y_1 \in \hat{B}_{u \setminus w}$ and $y_2 \in \hat{B}_{u \setminus w}$. There does not exist a vertex $m \in M_{u \setminus w}$ that is adjacent to y_1 and y_2 because otherwise, $y_1 x_1 w x_2 y_2 m y_1$ is a 6-hole. It follows that there exists a vertex $m_1 \in \hat{M}_{u \setminus w}$ adjacent to y_1 but not adjacent to y_2 , and there exists a vertex $m_2 \in \hat{M}_{u \setminus w}$ adjacent to y_2 but not adjacent to y_1 . Then for any $x \in \tilde{B}$, it holds that vertices m_1, m_2, y_1, u, y_2 , and x induce a type-a subgraph.

Thus there exists a vertex $y \in \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$ that dominates every other vertex in $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$. Because $y \notin B^*$, it holds that there exists a vertex $x \in \tilde{B}$ that is not adjacent to y , implying x is not adjacent to $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$. $\square$

We are now prepared to establish the following result:

Theorem 5.1. *Let (M, B) be a tight corner in a moderately balanced graph G , and suppose that (M, B) does not contain a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Let $x \in \tilde{B}$ such that x is not adjacent to $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$. If x is incident to the unique chord of a uniquely-chorded cycle, then (M, B) is a type-2 tight corner.*

Proof. Let us show that (M, B) is a type-2 tight corner. Define

$$M_1 = N_M(U) \cap N_M(W) \quad \text{and} \quad M_2 = M \setminus M_1,$$

and define

$$\hat{B}_1 = U \cup W, \quad \hat{B}_3 = \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}, \quad \text{and} \quad \hat{B}_2 = \hat{B} \setminus (\hat{B}_1 \cup \hat{B}_2).$$

Note that from Lemma 5.2 and the fact that x is not adjacent to $\hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u} = \hat{B}_3$, it holds that x is adjacent to a vertex in $\hat{B} \setminus (B^* \cup \hat{B}_3)$, and therefore $\hat{B}_2 \neq \emptyset$ and $M_1 \neq \emptyset$. Clearly $\hat{B}_1, \hat{B}_3 \neq \emptyset$ and $M_2 \neq \emptyset$, and therefore $\{\hat{B}_1, \hat{B}_2, \hat{B}_3\}$ is a partition of $\hat{B}$ and $\{M_1, M_2\}$ is a partition of M .

Next let us briefly verify the four properties of a type-2 tight corner:

1. Let $y \in B_1$. It holds that $\tilde{B} \subseteq N(y)$ because $B_1 = U \cup W \subseteq B^*$, and it clearly holds that $M_1 = N_M(U) \cap N_M(W) \subseteq N(y)$. So $\tilde{B} \cup M_1 \subseteq N(y)$, and thus the first property holds.
2. The graph $G[A \cup M_2]$ is connected because every vertex in M_2 is adjacent to A . Because $\hat{M}_{u \setminus w} \cup \hat{M}_{w \setminus u} \subseteq M_2$, it holds that every vertex in $\hat{B}_3 = \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus u}$ is adjacent to M_2 . So $G[A \cup M_2 \cup \hat{B}_3]$ is connected, and thus the second property holds.
3. Let $y \in \hat{B}_2$. Then either $y \in D$ or $y \in \hat{B} \setminus (B^* \cup \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus v})$. Suppose $y \in D$. By Lemma 5.3, it holds that $N_M(y) \subseteq N_M(U) \cap N_M(W) = M_1$, yielding y is not adjacent to M_2 . If $y \in \hat{B} \setminus (B^* \cup \hat{B}_{u \setminus w} \cup \hat{B}_{w \setminus v})$, then y is only adjacent to vertices in $N_{\hat{M}}(U) \cap N_{\hat{M}}(W) \subseteq M_1$, yielding y is not adjacent to M_2 , and thus the third property holds.
4. Because x is not adjacent to $\hat{B}_3$, the fourth property holds.

$\square$

5.5 Type-3 tight corners

Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. In this subsection we show that such a tight corner, under a certain condition, is a type-3 tight corner, see Theorem 5.2 below.

Let us establish some preliminaries.

Lemma 5.5. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Then there exist vertices $u, w \in B^*$ such that*

1. $\hat{M} \subseteq N(u)$,
2. w is adjacent to $\tilde{M} \setminus N(u)$, and
3. w is not adjacent to every vertex in $\hat{M}$.

Proof. Let $D = \{y \in B^* : \hat{M} \subseteq N(y)\}$. Because (M, B) is not a type-1 tight corner, there exist vertices $u \in D$ and $w \in B^* \setminus D$ such that w is adjacent to $\tilde{M} \setminus N(u)$, establishing the desired result. $\square$

Let us introduce some definitions that we will just use for the remainder of the subsection. Let $u, w \in B^*$ be as given in the setup of Lemma 5.5. Define $\hat{M}_{1w}$ to be vertices in $\hat{M}$ that are not adjacent to w , and define $\hat{B}_{1w}$ to be the vertices in $\hat{B} \setminus B^*$ that are adjacent to $\hat{M}_{1w}$. Note that $\hat{M}_{1w} \neq \emptyset$, and hence $\hat{B}_{1w} \neq \emptyset$. Also define the vertex sets $\hat{M}_{2w} := \hat{M} \setminus \hat{M}_{1w}$ and $\hat{B}_{2w} := \hat{B} \setminus (B^* \cup \hat{B}_{1w})$. (In Lemma 5.7 below, we establish a condition that ensures $\hat{B}_{2w} \neq \emptyset$, implying $\hat{M}_{2w} \neq \emptyset$.)

Lemma 5.6. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Then the following statements hold.*

1. If $y_1 \in \hat{B}_{1w}$ and $y_2 \in \hat{B}_{1w}$ are not comparable relative to $\tilde{B}$, then $N_{\tilde{B}}(y_1) \cap N_{\tilde{B}}(y_2) = \emptyset$.
2. If $y_1 \in \hat{B}_{1w}$ strictly dominates $y_2 \in \hat{B}_{1w}$ relative to $\tilde{B}$, then y_1 strictly dominates y_2 relative to $\hat{M}_{1w}$.
3. Suppose that $\hat{B}_{2w} \neq \emptyset$. Let $y_1 \in \hat{B}_{1w}$ and $y_2 \in \hat{B}_{2w}$. If $N_{\tilde{B}}(y_1) \cap N_{\tilde{B}}(y_2) \neq \emptyset$, then it holds that y_1 dominates y_2 relative to $\tilde{B}$.

Proof. Let us consider the first part of the lemma:

Part 1. Let $x_1 \in N_{\tilde{B}}(y_1) \setminus N(y_2)$ and $x_2 \in N_{\tilde{B}}(y_2) \setminus N(y_1)$. It holds that $N_{\hat{M}_{1w}}(y_1) \cap N_{\hat{M}_{1w}}(y_2) = \emptyset$ because otherwise, for $\hat{m} \in N_{\hat{M}_{1w}}(y_1) \cap N_{\hat{M}_{1w}}(y_2)$, we have $y_1 x_1 w x_2 y_2 \hat{m} y_1$ is a 6-hole. Let $m_1 \in N_{\hat{M}_{1w}}(y_1) \setminus N(y_2)$ and $m_2 \in N_{\hat{M}_{1w}}(y_2) \setminus N(y_1)$. We see that $N_{\tilde{B}}(y_1) \cap N_{\tilde{B}}(y_2) = \emptyset$ because if not, then for any vertex $x \in N_{\tilde{B}}(y_1) \cap N_{\tilde{B}}(y_2)$, it holds that vertices m_1, m_2, y_1, u, y_2 , and x induce a type-a subgraph.

Part 2. Let $x_1 \in N_{\tilde{B}}(y_1) \setminus N(y_2)$ and $x_2 \in N_{\tilde{B}}(y_2) \cap N(y_1)$. Let $\tilde{m} \in N_{\tilde{M}}(w)$. For the sake of contradiction, suppose that there exists a vertex $\hat{m} \in N_{\hat{M}_{1w}}(y_2) \setminus N(y_1)$. Then vertices $\tilde{m}$, $\hat{m}$, y_1 , w , y_2 , x_1 , and x_2 induce a type-b subgraph.

Part 3. Let $x_1 \in \tilde{B}$ be a neighbor of y_1 and y_2 , let $\tilde{m} \in \tilde{M}$ be a neighbor of w , and let $m_1 \in \hat{M}_{1w}$ be a neighbor of y_1 . For the sake of contradiction, suppose that there exists a vertex $x_2 \in \tilde{B}$ that is a neighbor of y_2 , but not a neighbor of y_1 . Then vertices $\tilde{m}$, m_1 , y_2 , w , y_1 , x_2 , and x_1 induce a type-b subgraph. $\square$

Lemma 5.7. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Also suppose that $xy \in E(\tilde{B}, \hat{B}_{1w})$ is the unique chord of a cycle in G , where $y \in \hat{B}_{1w}$ does not dominate any vertex in $\hat{B}_{1w}$ relative to $\tilde{B}$. Then x has a neighbor in $\hat{B}_{2w}$.*

Proof. For the sake of contradiction, suppose that x is not adjacent to $\hat{B}_{2w}$. To obtain a contradiction, it is sufficient to show, by Lemma 5.2, that every vertex $\hat{y} \in N(x) \setminus \{y\}$ dominates y relative to $\tilde{B}$. Let $\hat{y} \in N(x) \setminus \{y\}$. Then either $\hat{y} \in B^*$ or $\hat{y} \in \hat{B}_{1w}$. If $\hat{y} \in B^*$, then $\hat{y}$ dominates y relative to $\tilde{B}$ by the definition of B^* . If $\hat{y} \in \hat{B}_{1w}$, then part 1 of Lemma 5.6 implies $\hat{y}$ dominates y relative to $\tilde{B}$. $\square$

Lemma 5.8. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Also suppose that $\hat{M}_{2w} \neq \emptyset$. Then for all $y \in B^* \cup \hat{B}_{1w}$, it holds that $\hat{M}_{2w} \subseteq N(y)$.*

Proof. Let $y \in B^* \cup \hat{B}_{1w}$, and notice that if $y = u$ or $y = w$, then the result is immediate; accordingly, suppose that $y \neq u$ and $y \neq w$. Let $\tilde{m} \in N_{\tilde{M}}(w) \setminus N(u)$. For the sake of contradiction, suppose that there exists a vertex $m_2 \in \hat{M}_{2w} \setminus N(y)$.

Case 1. Suppose that $y \in B^*$. Let $m_1 \in \hat{M}_{1w}$ and $y_1 \in \hat{B}_{1w}$ such that m_1 is adjacent to y_1 . Also let $x_1 \in \tilde{B} \setminus N(y_1)$ and $x_2 \in N_{\tilde{B}}(y_1)$. If y is neither adjacent to $\tilde{m}$ nor adjacent to m_1 , then vertices $\tilde{m}$, m_1 , y , w , y_1 , x_1 , and x_2 induce a type-b subgraph. If y is adjacent to $\tilde{m}$, then for any $x \in \tilde{B}$, it holds that vertices $\tilde{m}$, m_2 , y , w , u , and x_2 induce a type-a subgraph. If y is adjacent to m_1 , then for $x \in \tilde{B}$, it holds that vertices m_1 , m_2 , y , u , w , and x induce a type-a subgraph.

Case 2. Suppose that $y \in \hat{B}_{1w}$, and let $m_1 \in \hat{M}_{1w}$ such that m_1 is adjacent to y . Then for any $x \in \tilde{B}$, it holds that vertices m_2 , m_1 , w , u , y , and x induce a type-a subgraph. $\square$

Let us introduce some more definitions that we will just use for the remainder of this subsection. For $I \subseteq \tilde{B}$, define the (possibly empty) vertex set $D_I := \{y \in \hat{B}_{1w} : D_I \subseteq N(y)\}$.

Lemma 5.9. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Then there exists a nonempty vertex set $J \subseteq \tilde{B}$ such that:*

1. no vertex in $\hat{B}_{1w} \setminus D_J$ is adjacent to J and $\tilde{B} \setminus J$,
2. every vertex in $\tilde{B} \setminus J$ is adjacent to $\hat{B}_{1w} \setminus J$, and

3. there exists a vertex $x \in J$ that is not adjacent to $\hat{B}_{1w} \setminus D_J$.

Proof. We consider two cases:

Case 1. Suppose that there exists a vertex in $\tilde{B}$ that is not adjacent to $\hat{B}_{1w}$. Then the desired result follows immediately by taking $J = \tilde{B}$ (implying $D_J = \emptyset$).

Case 2. Suppose that every vertex in $\tilde{B}$ is adjacent to $\hat{B}_{1w}$. Let us take

$$\begin{aligned} J \in \operatorname{argmin}_{I \subseteq \tilde{B}} |I| \\ \text{s.t.} \quad & \text{No vertex in } \hat{B}_{1w} \setminus D_I \text{ is adjacent to } I \text{ and } \tilde{B} \setminus I \\ & \text{Every vertex in } \tilde{B} \setminus I \text{ is adjacent to } \hat{B}_{1w} \setminus D_I \\ & I \neq \emptyset. \end{aligned} \tag{1}$$

We first show that (1) is feasible. Let $\hat{y} \in \hat{B}_{1w}$ such that no vertex in $\hat{B}_{1w}$ strictly dominates $\hat{y}$ relative to $\tilde{B}$, and let $I = \{y \in \hat{B}_{1w} : N_{\tilde{B}}(y) = N_{\tilde{B}}(\hat{y})\}$. Observe, by the first part of Lemma 5.6, it holds that $\hat{y}$ dominates every vertex in $\hat{B}_{1w}$ adjacent to $N_{\tilde{B}}(\hat{y})$. Also observe that every vertex in $\tilde{B} \setminus I$ is adjacent to $\hat{B}_{1w} \setminus D_I$ by the supposition that every vertex in $\tilde{B}$ is adjacent to $\hat{B}_{1w}$. It follows that I is feasible for (1).

It remains to show that J contains a vertex that is not adjacent to $\hat{B}_{1w} \setminus D_J$. Suppose that every vertex in J is adjacent to $\hat{B}_{1w} \setminus D_J$. Let $\hat{y} \in \hat{B}_{1w} \setminus D_J$ such that $\hat{y}$ is adjacent to J and no vertex in $\hat{B}_{1w} \setminus D_J$ strictly dominates $\hat{y}$ relative to J . Then for $\hat{I} = N_J(\hat{y}) = N_{\tilde{B}}(\hat{y})$, it holds that $D_{\hat{I}} = D_J \cup \{y \in \hat{B}_{1w} : N_J(y) = N_J(\hat{y})\}$. Observe, by the first part of Lemma 5.6, it holds that $\hat{y}$ dominates every vertex in $\hat{B}_{1w} \setminus D_{\hat{I}}$ adjacent to $\hat{I}$. Also observe that every vertex in $J \setminus \hat{I}$ is adjacent to $\hat{B}_{1w} \setminus D_{\hat{I}}$ by the supposition that every vertex in J is adjacent to $\hat{B}_{1w}$, and every vertex in $\tilde{B} \setminus J$ is adjacent to $\hat{B}_{1w} \setminus D_{\hat{I}}$ by the definition of J . Hence $\hat{I}$ is feasible for (1), but $|\hat{I}| < |J|$, a contradiction. $\square$

And for the remainder of the subsection, let $J \subseteq \tilde{B}$ be as given in the setup of Lemma 5.9. We are now prepared to establish the following result:

Theorem 5.2. *Let (M, B) be a tight corner in a moderately balanced graph G . Suppose that (M, B) is not a type-1 tight corner and that (M, B) contains a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. Let $x \in J \setminus N_{\tilde{B}}(\hat{B}_{1w} \setminus D_J)$, and let $y \in N(x)$ such that y does not strictly dominate any vertex in $N(x)$ relative to $\tilde{B}$. If xy is the unique chord of a cycle, then (M, B) is a type-3 tight corner.*

Proof. Let us show that (M, B) is a type-3 tight corner. For the partition $\{M_1, M_2\}$ of M , let us take $M_1 = \hat{M}_{2w}$ and $M_2 = \tilde{M} \cup \hat{M}_{1w}$. (Clearly $M_2 \neq \emptyset$, and we verify that $M_1 \neq \emptyset$ below.) We consider two cases:

Case 1. Suppose that there exists a vertex in $\tilde{B}$ that is not adjacent to $\hat{B}_{1w}$. Then $J = \tilde{B}$. Define

$$\tilde{B}_1 = \tilde{B} \setminus N(\hat{B}_{1w}) \quad \text{and} \quad \tilde{B}_2 = N_{\tilde{B}}(\hat{B}_{1w}),$$

and define

$$\hat{B}_1 = B^*, \quad \hat{B}_2 = N_{\tilde{B} \setminus B^*}(\tilde{B}_1), \quad \text{and} \quad \hat{B}_3 = \hat{B} \setminus (\hat{B}_1 \cup \hat{B}_2).$$

Note that the vertex sets $\tilde{B}_1$, $\tilde{B}_2$, $\hat{B}_1$, and $\hat{B}_3$ are nonempty. And $\hat{B}_2 \neq \emptyset$ because if not, then $y \in B^*$, contradicting Lemma 5.2. Because $\hat{B}_2 \subseteq \hat{B}_{2w}$, it follows that $M_1 = \hat{M}_{2w} \neq \emptyset$. Thus we have partitions of M , $\hat{B}$, and $\tilde{B}$.

Let us verify the four properties of a type-3 corner:

1. Let $y \in \hat{B}_1 = B^*$. Then $M_1 \cup \tilde{B}_1 \subseteq \hat{M}_{2w} \cup \tilde{B} \subseteq N(y)$, where the second inclusion follows from Lemma 5.8 and the definition of B^* , and thus the first property holds.
2. First observe that $G[A \cup M_2 \cup \hat{B}_{1w} \cup \tilde{B}_2]$ is connected because $G[A]$ is connected, every vertex in $\hat{M}_2 = \tilde{M} \cup \hat{M}_{1w}$ is adjacent to a vertex in A , every vertex in $\hat{B}_{1w}$ is adjacent to $\hat{M}_{1w}$, and every vertex in $\tilde{B}_2 = N_{\tilde{B}}(\hat{B}_{1w})$ is adjacent to $\hat{B}_{1w}$. Accordingly, to establish the second property, it is sufficient to show that every vertex in $\hat{B}_3 \setminus \hat{B}_{1w}$ is adjacent to $\tilde{B}_2$, and this holds because all vertices in $\hat{B}$ that are adjacent to $\tilde{B}_1$ are given by $\hat{B}_1 \cup \hat{B}_2$. Thus the second property holds.
3. From the definition of $\tilde{B}_1$ and $\hat{B}_2$, it follows that $\hat{B}_2 \subseteq \hat{B}_{2w}$, and hence no vertex in $\hat{B}_2$ is adjacent to $\tilde{M} \cup \hat{M}_{1w} = M_2$. It remains to show that no vertex in $\hat{B}_2$ is adjacent to $\tilde{B}_2$. This follows from Lemma 5.6 and the fact that every vertex in $\hat{B}_2$ is adjacent to a vertex in $\tilde{B}$ that no vertex in $\hat{B}_{1w}$ is adjacent to. Thus the third property holds.
4. The fourth property follows from the definition of the partition of $\hat{B}$.

Case 2. Suppose that every vertex in $\tilde{B}$ is adjacent to $\hat{B}_{1w}$. Define

$$\tilde{B}_1 = J \setminus N(\hat{B}_{1w} \setminus D_J) \quad \text{and} \quad \tilde{B}_2 = \tilde{B} \setminus \tilde{B}_1,$$

and define

$$\hat{B}_1 = B^* \cup D_J, \quad \hat{B}_2 = N_{\hat{B} \setminus \hat{B}_1}(\tilde{B}_1), \quad \text{and} \quad \hat{B}_3 = \hat{B} \setminus (\hat{B}_1 \cup \hat{B}_2).$$

Note that the vertex sets $\tilde{B}_1$, $\tilde{B}_2$, $\hat{B}_1$, and $\hat{B}_3$ are nonempty. Let us verify that $\hat{B}_2 \neq \emptyset$. Suppose that $\hat{B}_2 = \emptyset$. Then $y \in D_J \subseteq \hat{B}_{1w}$, but this contradicts Lemma 5.7. Thus $\hat{B}_2 \neq \emptyset$, and because $\hat{B}_2 \subseteq \hat{B}_{2w}$, it holds that $M_1 \neq \emptyset$. Therefore we have partitions of M , $\hat{B}$, and $\tilde{B}$.

Let us verify the four properties of a type-3 corner:

1. Let $y \in \hat{B}_1 = B^* \cup D_J$. If $y \in B^*$, then $M_1 \cup \tilde{B}_1 \subseteq N(y)$ from the same argument given in Case 1. If $y \in D_J$, then y is adjacent to every vertex in $J \supseteq \tilde{B}_1$, and from Lemma 5.8, it holds that $M_1 \subseteq N(y)$. Thus the first property holds.
2. It follows that $G[A \cup M_2 \cup (\hat{B}_{1w} \setminus D_J)]$ is connected from an argument similar to the argument given in Case 1. Next we show that $G[A \cup M_2 \cup (\hat{B}_{1w} \setminus D_J) \cup \tilde{B}_2]$ is connected. Observe that $\tilde{B}_2 = (\tilde{B} \setminus J) \cup N_J(\hat{B}_{1w} \setminus D_J)$. By the definition of J , every vertex in $\tilde{B} \setminus J$ is adjacent to $\hat{B}_{1w} \setminus D_J$. And clearly every vertex in $N_J(\hat{B}_{1w} \setminus D_J)$ is adjacent to $\hat{B}_{1w} \setminus D_J$. Thus $G[A \cup M_2 \cup (\hat{B}_{1w} \setminus D_J) \cup \tilde{B}_2]$ is connected. To establish the desired result, it is sufficient to show that every vertex in $\hat{B}_3 \setminus \hat{B}_{1w}$ is adjacent to $\tilde{B}_2$, and this follows from the fact that all vertices in $\hat{B}$ that are adjacent to $\tilde{B}_1$ are given by $\hat{B}_1 \cup \hat{B}_2$. Thus the second property holds.

3. Because D_J contains the only vertices in $\hat{B}_{1w}$ that are adjacent to $\tilde{B}_1$, it follows that $\hat{B}_2 \subseteq \hat{B}_{2w}$. Hence no vertex in $\hat{B}_2$ is adjacent to $\tilde{M} \cup \hat{M}_{1w} = M_2$. And from part 3 of Lemma 5.6, it follows that $N_{\tilde{B}}(\hat{B}_2) \subseteq \tilde{B}_1$, and hence no vertex in $\hat{B}_2$ is adjacent to $\tilde{B}_2$. Thus the third property holds.
4. The fourth property is immediate from the definition of the partition of $\hat{B}$.

□

5.6 Proof Theorem 1.4

We are now prepared to prove Theorem 1.4:

Proof of Theorem 1.4. Suppose that every edge in $E(\tilde{B}, \hat{B})$ is the unique chord of a cycle in G . To obtain a contradiction, we show that there exists a nonempty subset $U \subseteq B$ and tight corner (M', B') in $G \setminus U$ such that every edge in $E(\tilde{B}', \hat{B}')$ is the unique chord of a cycle in $G \setminus U$. Indeed, if this statement holds, then we can iteratively remove vertex sets from the graph, while maintaining the property that graph contains a corner, eventually contradicting the fact that the graph is finite.

We consider three cases. First, we consider the case in which (M, B) is a type-1 tight corner. Second, we consider the case in which there does not exist a vertex y such that $\hat{M} \subseteq N(y)$. And because every edge in $E(\tilde{B}, \hat{B})$ is the unique chord of a cycle, the conditions of Theorem 5.1 are met, and thus in this second case, (M, B) is a type-2 tight corner. Lastly, we consider the case in which (M, B) is not a type-1 tight corner and there exists a vertex $y \in B^*$ such that $\hat{M} \subseteq N(y)$. And because every edge in $E(\tilde{B}, \hat{B})$ is the unique chord of a cycle, the conditions of Theorem 5.2 are met, and thus in this last case, (M, B) is a type-3 tight corner.

Each of the cases proceed in a similar spirit:

Case 1. Suppose that (M, B) is a type-1 tight corner, and let $\{B_1^*, B_2^*\}$ be the corresponding partition of B^* . Let us take $U = B_1^*$:

First note that each vertex $y \in B_1^*$ dominates every vertex $\hat{y} \in \hat{B} \setminus B_1^*$ because if $\hat{y} \in B_2^*$, then $N(\hat{y}) = \tilde{B} \cup N_M(\hat{y}) \subseteq N(y)$, and if $\hat{y} \in \hat{B} \setminus B^*$, then $N(\hat{y}) \subseteq \tilde{B} \cup \hat{M} \subseteq N(y)$. It follows from Remark 2.1 that every edge in $E(\tilde{B}, \hat{B} \setminus B_1^*)$ is the unique chord of a cycle in $G \setminus B_1^*$. Hence, by Theorem 1.2, every minimal vertex in $B \setminus B_1^*$ is separating in $G \setminus B_1^*$.

Let $x \in \tilde{B}$, and let $B_x \subseteq B \setminus B_1^*$ be the vertex of the component in $G[B \setminus B_1^*]$ containing x . From Lemma 5.2, it holds that x is adjacent to a vertex in $\hat{B} \setminus B^* \subseteq \hat{B} \setminus B_1^*$, and hence, in $G \setminus B_1^*$, it holds that M is a minimal separator for A and B_x . Further, in $G \setminus B_1^*$, we have that $d(x, A) \geq 3$ and that every minimal vertex in $B_x \subseteq B \setminus B_1^*$ is separating. From Theorem 4.1, there exists a tight corner (M', B') in $G \setminus B_1^*$ satisfying $B' \subseteq B_x$. Because $E(\tilde{B}', \hat{B}') \subseteq E(\tilde{B}, \hat{B} \setminus B_1^*)$, it holds that every edge in $E(\tilde{B}', \hat{B}')$ is the unique chord of a cycle.

Case 2. Suppose that (M, B) is a type-2 tight corner, and let $\{\hat{B}_1, \hat{B}_2, \hat{B}_3\}$ and $\{M_1, M_2\}$ be the corresponding partitions of $\hat{B}$ and M . Let us take $U = \hat{B}_1$.

From the first and third property of a type-2 tight corner, it holds that each vertex in $\hat{B}_1$ dominates every vertex in $\hat{B}_2$, implying every edge in $E(\tilde{B}, \hat{B}_2)$ is the unique chord of a

cycle in $G \setminus \hat{B}_1$. Hence, by Theorem 1.2, every minimal vertex in $\hat{B}_2 \cup N_{\hat{B}}(\hat{B}_2)$ is a separating vertex in $G \setminus \hat{B}_1$.

By the fourth-property of a type-2 corner and Lemma 5.2, it holds that $N_{\hat{B}}(\hat{B}_2) \setminus N(\hat{B}_3)$ is nonempty. Let $x \in N_{\hat{B}}(\hat{B}_2) \setminus N(\hat{B}_3)$, and let B_x be the component in $G[\hat{B}_2 \cup (N_{\hat{B}}(\hat{B}_2) \setminus N(\hat{B}_3))]$ containing x . Then, in $G \setminus \hat{B}_1$, because (M, B) is a type-2 corner, it holds that $M_1 \cup N_{\hat{B}}(\hat{B}_3)$ is a pseudominimal separator for $A \cup M_2 \cup \hat{B}_3$ and B_x . Further, in $G \setminus \hat{B}_1$, we have that $d(x, A \cup M_2 \cup \hat{B}_3) \geq 3$ and that every vertex in $B_x \subseteq \hat{B}_2 \cup (N_{\hat{B}}(\hat{B}_2) \setminus N(\hat{B}_3))$ is separating. From Theorem 4.1, there exists a tight corner (M', B') in $G \setminus \hat{B}_1$ satisfying $B' \subseteq B_x$. Because $E(\tilde{B}', \hat{B}') \subseteq E(\tilde{B}, \hat{B}_2)$, it holds that every edge in $E(\tilde{B}', \hat{B}')$ is the unique chord of a cycle in $G \setminus \hat{B}_1$.

Case 3. Suppose that (M, B) is a type-3 tight corner. Let $\{M_1, M_2\}$, $\{\hat{B}_1, \hat{B}_2, \hat{B}_3\}$, and $\{\tilde{B}_1, \tilde{B}_2\}$ be the corresponding partitions of M , $\hat{B}$, and $\tilde{B}$, respectively. Let us show that the claim holds under the choice $U = \hat{B}_1$.

From the first and third property for a type-3 corner, it holds that each vertex in $\hat{B}_1$ dominates every vertex in $\hat{B}_2$, implying every edge in $E(\tilde{B}, \hat{B}_2)$ is the unique chord of a cycle in $G \setminus \hat{B}_1$. Hence, by Theorem 1.2, every minimal vertex in $\tilde{B}_1 \cup \hat{B}_2$ is a separating vertex in $G \setminus \hat{B}_1$.

Let $x \in \tilde{B}_1$, and let B_x be the component in $G[\tilde{B}_1 \cup \hat{B}_2]$ containing x . By the fourth property of a type-3 corner and Lemma 5.2, it holds that x is adjacent to a vertex in $\hat{B}_2$. In $G \setminus \hat{B}_1$, because (M, B) is a type-3 corner, it holds that M_1 is a pseudominimal separator for $A \cup M_2 \cup \hat{B}_3 \cup \hat{B}_2$ and B_x . Further, in $G \setminus \hat{B}_1$, it holds that $d(x, A \cup M_2 \cup \hat{B}_3 \cup \hat{B}_2) \geq 3$ and that every minimal vertex in $B_x \subseteq \tilde{B}_1 \cup \hat{B}_2$ is separating. Thus from Theorem 4.1, there exists a tight corner (M', B') in $G \setminus \hat{B}_1$ satisfying $B' \subseteq B_x$. Because $E(\tilde{B}', \hat{B}') \subseteq E(\tilde{B}, \hat{B}_2)$, every edge in $E(\tilde{B}', \hat{B}')$ is the unique chord of a cycle in $G \setminus \hat{B}_1$. $\square$

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